PAUL ZEITZ

THE ART AND CRAFT OF PROBLEM SOLVING

Third Edition

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Paul Zeitz

University of San Francisco

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The explorer is the person who is lost.

—Tim Cahill, *Jaguars Ripped My Flesh*

When detectives speak of the moment that a crime becomes theirs to investigate, they speak of "catching a case," and once caught, a case is like a cold: it clouds and consumes the catcher's mind until, like a fever, it breaks; or, if it remains unsolved, it is passed on like a contagion, from one detective to another, without ever entirely releasing its hold on those who catch it along the way.

—Philip Gourevitch, *A Cold Case*

Mathematics is the most social of the sciences.

—Ravi Vakil (personal communication)

Preface to the Third Edition

This is the first revision of *The Art and Craft of Problem Solving* in ten years, but in contrast to the second edition, the changes are modest. I have responded to readers' disparate wishes by fixing errors, removing boring problems, adding more easy problems, adding more difficult problems, and, especially, including more "folklore." The new Section 4.4 (mathematical games) is the only alteration visible in the table of contents. However, substantial additions have been made to the problems, with several new themes that allow the reader to explore a wide variety of topics, with varying degrees of guidance.

The new material was inspired by my experience over the past decade working with teachers and students in math circles. In these math circles, we investigate many different topics, ranging from simple magic tricks based on parity to understanding random variables to deep algebraic discoveries relying on the interplay between number theory and complex numbers. I have attempted to share hours and hours of fun, frustration, and discovery with these new problems.

I am still indebted to the people and institutions acknowledged in the last two prefaces. To this list, I'd like to add David Aukley, Bela Bajnok, Art Benjamin, Brian Conrey, Brianna Donaldson, Gordon Hamilton, Po-Shen Loh, Henri Picciotto, Richard Rusczyk, James Tanton, Alexander Zvonkin, the American Institute of Mathematics, the Banff International Research Station, and the Moscow Center for Continuous Mathematical Education. My supportive and good-humored wife and children give me the strength to complete projects like this while reminding me that there is more to life than mathematics.

In 2011, I was asked to write the forward to Stuyvesant High School's annual *Math Survey* magazine. My essay, "The Three Epigraphs,"[1] discussed the quotes at the front of the first two editions of this book, and my plan for the third edition's quote, if there were to be a third edition. I have kept my promise, and now there really are three epigraphs. To paraphrase the essay in a nutshell, these quotes reveal, to students, "the secret" to doing mathematics (or any other meaningful artistic endeavor, for that matter): Get lost, get obsessed, and get together! With these sentiments, I gratefully dedicate this edition to my students.

Paul Zeitz San Francisco, May 2016

[1] See `https://www.scribd.com/doc/93806707/Paul-Zeitz-Math-Survey-Foreward`

Preface to the Second Edition

This new edition of *The Art and Craft of Problem Solving* is an expanded and, I hope, improved version of the original work. There are several changes, including:

- A new chapter on geometry. It is long—as many pages as the combinatorics and number theory chapters combined—but it is merely an introduction to the subject. Experts are bound to be dissatisfied with the chapter's pace (slow, especially at the start) and missing topics (solid geometry, directed lengths, and angles, Desargues's theorem, the 9-point circle). But this chapter is for beginners; hence its title, "Geometry for Americans." I hope that it gives the novice problem solver the confidence to investigate geometry problems as aggressively as he or she might tackle discrete math questions.
- An expansion to the calculus chapter, with many new problems.
- More problems, especially "easy" ones, in several other chapters.

To accommodate the new material and keep the length under control, the problems are in a two-column format with a smaller font. But don't let this smaller size fool you into thinking that the problems are less important than the rest of the book. As with the first edition, the problems are the heart of the book. The serious reader should, at the very least, read each problem statement, and attempt as many as possible. To facilitate this, I have expanded the number of problems discussed in the Hints appendix, which now can be found online at `www.wiley.com/college/zeitz`.

I am still indebted to the people that I thanked in the preface to the first edition. In addition, I'd like to thank the following people:

- Jennifer Battista and Ken Santor at Wiley expertly guided me through the revision process, never once losing patience with my procrastination.
- Brian Borchers, Joyce Cutler, Julie Levandosky, Ken Monks, Deborah Moore-Russo, James Stein, and Draga Vidakovic carefully reviewed the manuscript, found many errors, and made numerous important suggestions.
- At the University of San Francisco, where I have worked since 1992, Dean Jennifer Turpin and Associate Dean Brandon Brown have generously supported my extracurricular activities, including approval of a sabbatical leave during the 2005–06 academic year which made this project possible.
- Since 1997, my understanding of problem solving has been enriched by my work with a number of local math circles and contests. The Mathematical Sciences Research Institute (MSRI) has sponsored much of this activity, and I am particularly indebted to MSRI officers Hugo Rossi, David Eisenbud, Jim Sotiros, and Joe Buhler. Others who have helped me tremendously include Tom Rike, Sam Vandervelde, Mark Saul, Tatiana Shubin, Tom Davis, Josh Zucker, and especially, Zvezdelina Stankova.

And last but not least, I'd like to continue my contrition from the first edition, and ask my wife and two children to forgive me for my sleep-deprived inattentiveness. I dedicate this book, with love, to them.

Paul Zeitz San Francisco, June 2006

Preface to the First Edition

Why This Book?

This is a book about mathematical problem solving for college-level novices. By this I mean bright people who know some mathematics (ideally, at least some calculus), who enjoy mathematics, who have at least a vague notion of proof, but who have spent most of their time doing exercises rather than problems.

An *exercise* is a question that tests the student's mastery of a narrowly focused technique, usually one that was recently "covered." Exercises may be hard or easy, but they are never puzzling, for it is always immediately clear how to proceed. Getting the solution may involve hairy technical work, but the path toward solution is always apparent. In contrast, a *problem* is a question that cannot be answered immediately. Problems are often open-ended, paradoxical, and sometimes unsolvable, and require investigation before one can come close to a solution. Problems and problem solving are at the heart of mathematics. Research mathematicians do nothing but open-ended problem solving. In industry, being able to solve a poorly defined problem is much more important to an employer than being able to, say, invert a matrix. A computer can do the latter, but not the former.

A good problem solver is not just more employable. Someone who learns how to solve mathematical problems enters the mainstream culture of mathematics; he or she develops great confidence and can inspire others. Best of all, problem solvers have fun; the adept problem solver knows how to play with mathematics, and understands and appreciates beautiful mathematics.

An analogy: The average (non-problem-solver) math student is like someone who goes to a gym three times a week to do lots of repetitions with low weights on various exercise machines. In contrast, the problem solver goes on a long, hard backpacking trip. Both people get stronger. The problem solver gets hot, cold, wet, tired, and hungry. The problem solver gets lost, and has to find his or her way. The problem solver gets blisters. The problem solver climbs to the top of mountains, sees hitherto undreamed of vistas. The problem solver arrives at places of amazing beauty, and experiences ecstasy that is amplified by the effort expended to get there. When the problem solver returns home, he or she is energized by the adventure, and cannot stop gushing about the wonderful experience. Meanwhile, the gym rat has gotten steadily stronger, but has not had much fun, and has little to share with others.

While the majority of American math students are not problem solvers, there does exist an elite problem solving culture. Its members were raised with math clubs, and often participated in math contests, and learned the important "folklore" problems and ideas that most mathematicians take for granted. This culture is prevalent in parts of Eastern Europe and exists in small pockets in the United States. I grew up in New York City and attended Stuyvesant High School, where I was captain of the math team, and consequently had a problem solver's education. I was and am deeply involved with problem solving contests. In high school, I was a member of the first USA team to participate in the International Mathematical Olympiad (IMO) and twenty years later, as a college professor, have coached several of the most recent IMO teams, including one which in 1994 achieved the only perfect performance in the history of the IMO.

But most people don't grow up in this problem solving culture. My experiences as a high school and college teacher, mostly with students who did not grow up as problem solvers, have convinced me that problem solving is something that is easy for any bright math student to learn. As a missionary for the problem solving culture, *The Art and Craft of Problem Solving* is a first approximation of my attempt to spread the gospel. I decided to write this book because I could not find any suitable text that worked for my students at the University of San Francisco. There are many nice books with lots of good mathematics out there, but I have found that mathematics itself is not enough. *The Art and Craft of Problem Solving* is guided by several principles:

- Problem solving can be taught and can be learned.
- Success at solving problems is crucially dependent on psychological factors. Attributes like confidence, concentration, and courage are vitally important.
- No-holds-barred investigation is at least as important as rigorous argument.
- The non-psychological aspects of problem solving are a mix of strategic principles, more focused tactical approaches, and narrowly defined technical tools.
- Knowledge of folklore (for example, the pigeonhole principle or Conway's Checker problem) is as important as mastery of technical tools.

Reading This Book

Consequently, although this book is organized like a standard math textbook, its tone is much less formal: it tries to play the role of a friendly coach, teaching not just by exposition, but by exhortation, example, and challenge. There are few prerequisites—only a smattering of calculus is assumed—and while my target audience is college math majors, the book is certainly accessible to advanced high school students and to people reading on their own, especially teachers (at any level).

The book is divided into two parts. Part I is an overview of problem-solving methodology, and is the core of the book. Part II contains four chapters that can be read independently of one another and outline algebra, combinatorics, number theory, and calculus from the problem solver's point of view.[2] In order to keep the book's length manageable, there is no chapter on geometry. Geometric ideas are diffused throughout the book, and concentrated in a few places (for example, Section 4.2). Nevertheless, the book is a bit light on geometry. Luckily, a number of great geometry books have already been written. At the elementary level, *Geometry Revisited* [4] and *Geometry and the Imagination* [16] have no equals.

The structure of each section within each chapter is simple: exposition, examples, and problems—lots and lots—some easy, some hard, some very hard. The purpose of the book is to teach problem solving, and this can only be accomplished by grappling with many problems, solving some and learning from others that not every problem is meant to be solved, and that any time spent thinking honestly about a problem is time well spent.

My goal is that reading this book and working on some of its 660 problems should be like the backpacking trip described above. The reader will definitely get lost for some of the time, and will get very, very sore. But at the conclusion of the trip, the reader will be toughened and happy and ready for more adventures.

And he or she will have learned a lot about mathematics—not a specific branch of mathematics, but mathematics, pure and simple. Indeed, a recurring theme throughout the book is the unity of mathematics. Many of the specific problem solving methods involve the idea of recasting from one branch of math to another; for example, a geometric interpretation of an algebraic inequality.

[2] To conserve pages, the second edition no longer uses formal "Part I" and "Part II" labels. Nevertheless, the book has the same logical structure, with an added chapter on geometry. For more information about how to read the book, see Section 1.4.

Teaching With This Book

In a one-semester course, virtually all of Part I should be studied, although not all of it will be mastered. In addition, the instructor can choose selected sections from Part II. For example, a course at the freshman or sophomore level might concentrate on Chapters 1–6, while more advanced classes would omit much of Chapter 5 (except the last section) and Chapter 6, concentrating instead on Chapters 7 and 8.

This book is aimed at beginning students, and I don't assume that the instructor is expert, either. The *Instructor's Resource Manual* contains solution sketches to most of the problems as well as some ideas about how to teach a problem solving course. For more information, please visit `www.wiley.com/college/zeitz`.

Acknowledgments

Deborah Hughes Hallet has been the guardian angel of my career for nearly twenty years. Without her kindness and encouragement, this book would not exist, nor would I be a teacher of mathematics. I owe it to you, Deb. Thanks!

I have had the good fortune to work at the University of San Francisco, where I am surrounded by friendly and supportive colleagues and staff members, students who love learning, and administrators who strive to help the faculty. In particular, I'd like to single out a few people for heartfelt thanks:

- My dean, Stanley Nel, has helped me generously in concrete ways, with computer upgrades and travel funding. But more importantly, he has taken an active interest in my work from the very beginning. His enthusiasm and the knowledge that he supports my efforts have helped keep me going for the past four years.
- Tristan Needham has been my mentor, colleague, and friend since I came to USF in 1992. I could never have finished this book without his advice and hard labor on my behalf. Tristan's wisdom spans the spectrum from the tiniest LaTeX details to deep insights about the history and foundations of mathematics. In many ways that I am still just beginning to understand, Tristan has taught me what it means to really understand a mathematical truth.
- Nancy Campagna, Marvella Luey, Tonya Miller, and Laleh Shahideh have generously and creatively helped me with administrative problems so many times and in so many ways that I don't know where to begin. Suffice to say that without their help and friendship, my life at USF would often have become grim and chaotic.
- Not a day goes by without Wing Ng, our multitalented department secretary, helping me to solve problems involving things such as copier misfeeds to software installation to page layout. Her ingenuity and altruism have immensely enhanced my productivity.

Many of the ideas for this book come from my experiences teaching students in two vastly different arenas: a problem-solving seminar at USF and the training program for the USA team for the IMO. I thank all of my students for giving me the opportunity to share mathematics.

My colleagues in the math competitions world have taught me much about problem solving. In particular, I'd like to thank Titu Andreescu, Jeremy Bem, Doug Jungreis, Kiran Kedlaya, Jim Propp, and Alexander Soifer for many helpful conversations.

Bob Bekes, John Chuchel, Dennis DeTurk, Tim Sipka, Robert Stolarsky, Agnes Tuska, and Graeme West reviewed earlier versions of this book. They made many useful comments and found many errors. The book is much improved because of their careful reading. Whatever errors remain, I of course assume all responsibility.

This book was written on a Macintosh computer, using LaTeX running on the wonderful Textures program, which is miles ahead of any other TeX system. I urge anyone contemplating writing a book using TeX or LaTeX to consider this program (www.bluesky.com). Another piece of software that helped me immensely was Eric Scheide's indexer program, which automates much of the LaTeX indexing process. His program easily saved me a week's tedium. Contact scheide@usfca.edu for more information.

Ruth Baruth, my editor at Wiley, has helped me transform a vague idea into a book in a surprisingly short time, by expertly mixing generous encouragement, creative suggestions, and gentle prodding. I sincerely thank her for her help, and look forward to more books in the future.

My wife and son have endured a lot during the writing of this book. This is not the place for me to thank them for their patience, but to apologize for my neglect. It is certainly true that I could have gotten a lot more work done, and done the work that I did do with less guilt, if I didn't have a family making demands on my time. But without my family, nothing—not even the beauty of mathematics—would have any meaning at all.

Paul Zeitz San Francisco, November, 1998

Contents

1 What This Book Is About and How to Read It

1.1 “Exercises” vs. “Problems”

This is a book about mathematical problem solving. We make three assumptions about you, our reader:

- You enjoy math.
- You know high-school math pretty well, and have at least begun the study of “higher mathematics” such as calculus and linear algebra.
- You want to become better at solving math problems.

First, what is a *problem*? We distinguish between *problems* and *exercises*. An exercise is a question that you know how to resolve immediately. Whether you get it right or not depends on how expertly you apply specific techniques, but you don’t need to puzzle out what techniques to use. In contrast, a problem demands much thought and resourcefulness before the right approach is found. For example, here is an exercise.

Example 1.1.1 Compute 5436^3 without a calculator.

You have no doubt about how to proceed—just multiply, carefully. The next question is more subtle.

Example 1.1.2 Write

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}$$

as a fraction in lowest terms.

At first glance, it is another tedious exercise, for you can just carefully add up all 99 terms, and hope that you get the right answer. But a little investigation yields something intriguing. Adding the first few terms and simplifying, we discover that

$$\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3} &= \frac{2}{3},\\
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4} &= \frac{3}{4},\\
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5} &= \frac{4}{5},
\end{aligned}$$

which leads to the *conjecture* that for all positive integers n,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

So now we are confronted with a *problem*: is this conjecture true, and if so, how do we *prove* that it is true? If we are experienced in such matters, this is still a mere exercise, in the technique of mathematical induction (see page 42). But if we are not experienced, it is a problem, not an exercise. To solve it, we need to spend some time, trying out different approaches. The harder the problem is, the more time we need. Often the first approach fails. Sometimes the first dozen approaches fail!

Here is another question, the famous "Census-Taker Problem." A few people might think of this as an exercise, but for most, it is a problem.

Example 1.1.3 A census-taker knocks on a door, and asks the woman inside how many children she has and how old they are.

"I have three daughters, their ages are whole numbers, and the product of the ages is 36," says the mother.

"That's not enough information," responds the census-taker.

"I'd tell you the sum of their ages, but you'd still be stumped."

"I wish you'd tell me something more."

"Okay, my oldest daughter Annie likes dogs."

What are the ages of the three daughters?

After the first reading, it seems impossible—there isn't enough information to determine the ages. That's why it is a problem, and a fun one, at that. (The answer is at the end of this chapter, on page 11, if you get stumped.)

If the Census-Taker Problem is too easy, try this next one (see page 72 for solution):

Example 1.1.4 I invite 10 couples to a party at my house. I ask everyone present, including my wife, how many people they shook hands with. It turns out that everyone questioned—I didn't question myself, of course—shook hands with a different number of people. If we assume that no one shook hands with his or her partner, how many people did my wife shake hands with? (I did not ask myself any questions.)

A good problem is mysterious and interesting. It is mysterious, because at first you don't know how to solve it. If it is not interesting, you won't think about it much. If it is interesting, though, you will want to put a lot of time and effort into understanding it.

This book will help you to investigate and solve problems. If you are an inexperienced problem solver, you may often give up quickly. This happens for several reasons.

- You may just not know how to begin.
- You may make some initial progress, but then cannot proceed further.
- You try a few things, nothing works, so you give up.

An experienced problem solver, in contrast, is rarely at a loss for how to begin investigating a problem. He or she[1] confidently tries a number of approaches to get started. This may not solve the problem,

[1] We will henceforth avoid the awkward "he or she" construction by choosing genders randomly in subsequent chapters.

but some progress is made. Then more specific techniques come into play. Eventually, at least some of the time, the problem is resolved. The experienced problem solver operates on three different levels:

Strategy: Mathematical and psychological ideas for starting and pursuing problems.

Tactics: Diverse mathematical methods that work in many different settings.

Tools: Narrowly focused techniques and "tricks" for specific situations.

1.2 The Three Levels of Problem Solving

Some branches of mathematics have very long histories, with many standard symbols and words. Problem solving is not one of them.[2] We use the terms *strategy*, *tactics*, and *tools* to denote three different levels of problem solving. Since these are not standard definitions, it is important that we understand exactly what they mean.

A Mountaineering Analogy

You are standing at the base of a mountain, hoping to climb to the summit. Your first *strategy* may be to take several small trips to various easier peaks nearby, so as to observe the target mountain from different angles. After this, you may consider a somewhat more focused strategy, perhaps to try climbing the mountain via a particular ridge. Now the *tactical* considerations begin: how to actually achieve the chosen strategy. For example, suppose that strategy suggests climbing the south ridge of the peak, but there are snowfields and rivers in our path. Different tactics are needed to negotiate each of these obstacles. For the snowfield, our tactic may be to travel early in the morning, while the snow is hard. For the river, our tactic may be scouting the banks for the safest crossing. Finally, we move onto the most tightly focused level, that of *tools*: specific techniques to accomplish specialized tasks. For example, to cross the snowfield, we may set up a particular system of ropes for safety and walk with ice axes. The river crossing may require the party to strip from the waist down and hold hands for balance. These are all tools. They are very specific. You would never summarize, "To climb the mountain we had to take our pants off and hold hands," because this was a minor—though essential—component of the entire climb. On the other hand, strategic and sometimes tactical ideas are often described in your summary: "We decided to reach the summit via the south ridge and had to cross a difficult snowfield and a dangerous river to get to the ridge."

As we climb a mountain, we may encounter obstacles. Some of these obstacles are easy to negotiate, for they are mere exercises (of course this depends on the climber's ability and experience). But one obstacle may present a difficult miniature problem, whose solution clears the way for the entire climb. For example, the path to the summit may be easy walking, except for one 10-foot section of steep ice. Climbers call negotiating the key obstacle the *crux move*. We shall use this term for mathematical problems as well. A crux move may take place at the strategic, tactical, or tool level; some problems have several crux moves; many have none.

[2] In fact, there does not even exist a standard name for the theory of problem solving, although George Pólya and others have tried to popularize the term *heuristics* (see, for example, [24]).

From Mountaineering to Mathematics

Let's approach mathematical problems with these mountaineering ideas. When confronted with a problem, you cannot immediately solve it, for otherwise, it is not a problem but a mere exercise. You must begin a process of *investigation*. This investigation can take many forms. One method, by no means a terrible one, is to just randomly try whatever comes into your head. If you have a fertile imagination, and a good store of methods, and a lot of time to spare, you may eventually solve the problem. However, if you are a beginner, it is best to cultivate a more organized approach. First, think strategically. Don't try immediately to solve the problem, but instead think about it on a less-focused level. The goal of strategic thinking is to come up with a plan that may only barely have mathematical content, but which leads to an "improved" situation, not unlike the mountaineer's strategy, "If we get to the south ridge, it looks like we will be able to get to the summit."

Strategies help us get started, and help us continue. But they are just vague outlines of the actual work that needs to be done. The concrete tasks to accomplish our strategic plans are done at the lower levels of tactic and tool.

Here is an example that shows the three levels in action, from a 1926-Hungarian contest.

Example 1.2.1 Prove that the product of four consecutive natural numbers cannot be the square of an integer.

Solution: Our initial strategy is to familiarize ourselves with the statement of the problem, i.e., to *get oriented*. We first note that the question asks us to prove something. Problems are usually of two types—those that ask you to prove something and those that ask you to find something. The Census-Taker Problem (Example 1.1.3) is an example of the latter type.

Next, observe that the problem is asking us to prove that something *cannot* happen. We divide the problem into *hypothesis* (also called "the given") and *conclusion* (whatever the problem is asking you to find or prove). The hypothesis is:

Let n be a natural number.

The conclusion is:

$n(n+1)(n+2)(n+3)$ *cannot be the square of an integer.*

Formulating the hypothesis and conclusion isn't a triviality, since many problems don't state them precisely. In this case, we had to introduce some notation. Sometimes our choice of notation can be critical.

Perhaps we should focus on the conclusion: how do you go about showing that something cannot be a square? This strategy, trying to think about what would immediately lead to the conclusion of our problem, is called looking at the *penultimate step*.[3] Unfortunately, our imagination fails us—we cannot think of any easy criteria for determining when a number cannot be a square. So we try another strategy, one of the best for beginning just about any problem: *get your hands dirty*. We try plugging in some numbers to experiment with. If we are lucky, we may see a pattern. Let's try a few different values for n. Here's a table. We use the abbreviation $f(n) = n(n+1)(n+2)(n+3)$.

[3] The word "penultimate" means "next to last."

n	1	2	3	4	5	10
$f(n)$	24	120	360	840	1680	17160

Notice anything? The problem involves squares, so we are sensitized to look for squares. Just about everyone notices that the first two values of $f(n)$ are one less than a perfect square. A quick check verifies that additionally,

$$f(3) = 19^2 - 1, \quad f(4) = 29^2 - 1, \quad f(5) = 41^2 - 1, \quad f(10) = 131^2 - 1.$$

We confidently conjecture that $f(n)$ is one less than a perfect square for every n. Proving this conjecture is the penultimate step that we were looking for, *because a positive integer that is one less than a perfect square cannot be a perfect square* since the sequence $1, 4, 9, 16, \ldots$ of perfect squares contains no consecutive integers (the gaps between successive squares get bigger and bigger). Our new strategy is to prove the conjecture.

To do so, we need help at the tactical/tool level. We wish to prove that for each n, the product $n(n+1)(n+2)(n+3)$ is one less than a perfect square. In other words, $n(n+1)(n+2)(n+3)+1$ must be a perfect square. How to show that an algebraic expression is always equal to a perfect square? One tactic: *factor* the expression! We need to manipulate the expression, always keeping in mind our goal of getting a square. So we focus on putting parts together that are almost the same. Notice that the product of n and $n+3$ is "almost" the same as the product of $n+1$ and $n+2$, in that their first two terms are both $n^2 + 3n$. After regrouping, we have

$$[n(n+3)][(n+1)(n+2)] + 1 = (n^2+3n)(n^2+3n+2) + 1. \tag{1.2.1}$$

Rather than multiply out the two almost-identical terms, we introduce a little *symmetry* to bring squares into focus:

$$(n^2+3n)(n^2+3n+2) + 1 = \big((n^2+3n+1) - 1\big)\big((n^2+3n+1) + 1\big) + 1.$$

Now we use the "difference of two squares" factorization (a tool!) and we have

$$\begin{aligned}\big((n^2+3n+1) - 1\big)\big((n^2+3n+1) + 1\big) + 1 &= (n^2+3n+1)^2 - 1 + 1\\ &= (n^2+3n+1)^2.\end{aligned}$$

We have shown that $f(n)$ is one less than a perfect square for all integers n, namely

$$f(n) = (n^2+3n+1)^2 - 1,$$

and we are done. ■

Let us look back and analyze this problem in terms of the three levels. Our first strategy was orientation, reading the problem carefully and classifying it in a preliminary way. Then we decided on a strategy to look at the penultimate step that did not work at first, but the strategy of numerical experimentation led to a conjecture. Successfully proving this involved the tactic of factoring, coupled with a use of symmetry and the tool of recognizing a common factorization.

The most important level was strategic. Getting to the conjecture was the crux move. At this point the problem metamorphosed into an exercise! For even if you did not have a good tactical grasp, you could have muddled through. One fine method is *substitution*: Let $u = n^2 + 3n$ in equation (1.2.1). Then the right-hand side becomes $u(u + 2) + 1 = u^2 + 2u + 1 = (u + 1)^2$. Another method is to multiply out (ugh!). We have

$$n(n + 1)(n + 2)(n + 3) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1.$$

If this is going to be the square of something, it will be the square of the quadratic polynomial $n^2 + an + 1$ or $n^2 + an - 1$. Trying the first case, we equate

$$n^4 + 6n^3 + 11n^2 + 6n + 1 = (n^2 + an + 1)^2 = n^4 + 2an^3 + (a^2 + 2)n^2 + 2an + 1$$

and we see that $a = 3$ works; i.e., $n(n + 1)(n + 2)(n + 3) + 1 = (n^2 + 3n + 1)^2$. This was a bit less elegant than the first way we solved the problem, but it is a fine method. Indeed, it teaches us a useful *tool*: the method of *undetermined coefficients*.

1.3 A Problem Sampler

The problems in this book are classified into three large families: *recreational*, *c*ontest, and *open-ended*. Within each family, problems split into two basic kinds: problems *"to find"* and problems *"to prove."*[4] Problems "to find" ask for a specific piece of information, while problems "to prove" require a more general argument. Sometimes the distinction is blurry. For example, Example 1.1.4 above is a problem "to find," but its solution may involve a very general argument.

What follows is a descriptive sampler of each family.

Recreational Problems

Also known as "brain teasers," these problems usually involve little formal mathematics, but instead rely on creative use of basic strategic principles. They are excellent to work on, because no special knowledge is needed, and any time spent thinking about a recreational problem will help you later with more mathematically sophisticated problems. The Census-Taker Problem (Example 1.1.3) is a good example of a recreational problem. A gold mine of excellent recreational problems is the work of Martin Gardner, who edited the "Mathematical Games" department for *Scientific American* for many years. Many of his articles have been collected into books. Two of the nicest are perhaps [9] and [8].

1.3.1 A monk climbs a mountain. He starts at 8 AM and reaches the summit at noon. He spends the night on the summit. The next morning, he leaves the summit at 8 AM and descends by the same route that he used the day before, reaching the bottom at noon. Prove that there is a time between 8 AM and noon at which the monk was at exactly the same spot on the mountain on both days. (Notice that we do not specify anything about the speed that the monk travels. For example, he could race at 1000 miles per hour for the first few minutes, then sit still for hours, then travel backward, etc. Nor does the monk have to travel at the same speeds going up as going down.)

[4] These two terms are due to George Pólya [24].

1.3.2 You are in the downstairs lobby of a house. There are three switches, all in the "off" position. Upstairs, there is a room with a lightbulb that is turned off. One and only one of the three switches controls the bulb. You want to discover which switch controls the bulb, but you are only allowed to go upstairs once. How do you do it? (No fancy strings, telescopes, etc. allowed. You cannot see the upstairs room from downstairs. The lightbulb is a standard 100-watt bulb.)

1.3.3 You leave your house, travel one mile due south, then one mile due east, then one mile due north. You are now back at your house! Where do you live? There is more than one solution; find as many as possible.

Contest Problems

These problems are written for formal exams with time limits, often requiring specialized tools and/or ingenuity to solve. Several exams at the high-school and undergraduate levels involve sophisticated and interesting mathematics.

American High School Math Exam (AHSME) Taken by hundreds of thousands of self-selected high-school students each year, this multiple-choice test has questions similar to the hardest and most interesting problems on the SAT.[5]

American Invitational Math Exam (AIME) The top 2000 or so scorers on the AHSME qualify for this three-hour, 15-question test. Both the AHSME and AIME feature problems "to find," since these tests are graded by machine.

USA Mathematical Olympiad (USAMO) The top 150 AIME participants participate in this elite three-and-a-half-hour, five-question essay exam, featuring mostly challenging problems "to prove."[6]

American Regions Mathematics League (ARML) Every year, ARML conducts a national contest between regional teams of high-school students. Some of the problems are quite challenging and interesting, roughly comparable to the harder questions on the AHSME and AIME and the easier USAMO problems.

Other national and regional olympiads Many other nations conduct difficult problem solving contests. Eastern Europe in particular has a very rich contest tradition, including very recently China and Vietnam have developed very innovative and challenging examinations.

International Mathematical Olympiad (IMO) The top USAMO scorers are invited to a training program, which then selects the six-member USA team that competes in this international contest. It is a nine-hour, six-question essay exam, spread over two days.[7] The IMO began in 1959, and takes place in a different country each year. At first it was a small event restricted to Iron Curtain countries, but recently the event has become quite inclusive, with 75 nations represented in 1996.

[5] Recently, this exam has been replaced by the AMC-8, AMC-10, and AMC-12 exams, for different targeted grade levels.
[6] There now is an exam for younger students, the USA Junior Mathematical Olympiad (USAJMO).
[7] Starting in 1996, the USAMO adopted a similar format: six questions, taken during two sessions.

Putnam Exam The most important problem solving contest for American undergraduates, a 12-question, six-hour exam taken by several thousand students each December. The median score is often zero.

Problems in magazines A number of mathematical journals have problem departments, in which readers are invited to propose problems and/or mail in solutions. The most interesting solutions are published, along with a list of those who solved the problem. Some of these problems can be extremely difficult, and many remain unsolved for years. Journals with good problem departments, in increasing order of difficulty, are *Math Horizons, The College Mathematics Journal, Mathematics Magazine*, and *The American Mathematical Monthly*. All of these are published by the Mathematical Association of America. There is also a journal devoted entirely to interesting problems and problem solving, *Crux Mathematicorum*, published by the Canadian Mathematical Society.

Contest problems are very challenging. It is a significant accomplishment to solve a single such problem, even with no time limit. The samples below include problems of all difficulty levels.

1.3.4 (AHSME 1996) In the xy-plane, what is the length of the shortest path from $(0,0)$ to $(12,16)$ that does not go inside the circle $(x-6)^2+(y-8)^2=25$?

1.3.5 (AHSME 1996) Given that $x^2+y^2=14x+6y+6$, what is the largest possible value that $3x+4y$ can have?

1.3.6 (AHSME 1994) When n standard six-sided dice are rolled, the probability of obtaining a sum of 1994 is greater than zero and is the same as the probability of obtaining a sum of S. What is the smallest possible value of S?

1.3.7 (AIME 1994) Find the positive integer n for which

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 n \rfloor = 1994,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. (For example, $\lfloor \pi \rfloor = 3$.)

1.3.8 (AIME 1994) For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2-a_1, a_3-a_2, a_4-a_3, \ldots)$ whose nth term is $a_{n+1}-a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{94} = 0$. Find a_1.

1.3.9 (USAMO 1989) The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of six games with 12 distinct players.

1.3.10 (USAMO 1995) A calculator is broken so that the only keys that still work are the $\sin$, $\cos$, $\tan$, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

1.3.11 (IMO 1976) Determine, with proof, the largest number that is the product of positive integers whose sum is 1976.

1.3.12 (Russia 1996) A *palindrome* is a number or word that is the same when read forward and backward, for example, "176671" and "civic." Can the number obtained by writing the numbers from 1 to n in order (for some $n > 1$) be a palindrome?

1.3.13 (Putnam 1994) Let (a_n) be a sequence of positive reals such that, for all n, $a_n \le a_{2n} + a_{2n+1}$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

1.3.14 (Putnam 1994) Find the positive value of m such that the area in the first quadrant enclosed by the ellipse $x^2/9 + y^2 = 1$, the x-axis, and the line $y = 2x/3$ is equal to the area in the first quadrant enclosed by the ellipse $x^2/9 + y^2 = 1$, the y-axis, and the line $y = mx$.

1.3.15 (Putnam 1990) Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

Open-Ended Problems

These are mathematical questions that are sometimes vaguely worded, and possibly have no actual solution (unlike the two types of problems described above). Open-ended problems can be very exciting to work on, because you don't know what the outcome will be. A good open-ended problem is like a hike (or expedition!) in an uncharted region. Often partial solutions are all that you can get. (Of course, partial solutions are always OK, even if you know that the problem you are working on is a formal contest problem that has a complete solution.)

1.3.16 Here are the first few rows of *Pascal's Triangle*.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1, \end{array}$$

where the elements of each row are the sums of pairs of adjacent elements of the prior row. For example, $10 = 4 + 6$. The next row in the triangle will be

$$1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1.$$

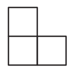
There are many interesting patterns in Pascal's Triangle. Discover as many patterns and relationships as you can, and prove as much as possible. In particular, can you somehow extract the Fibonacci numbers (see next problem) from Pascal's Triangle (or vice versa)? Another question: is there a pattern or rule for the *parity* (evenness or oddness) of the elements of Pascal's Triangle?

1.3.17 The *Fibonacci numbers* f_n are defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. For example, $f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,\ f_6 = 8,\ f_7 = 13,\ f_8 = 21$. Play around with this sequence; try to discover as many patterns as you can, and try to prove your conjectures as best as you can. In particular, look at this amazing fact: for $n \geq 0$,

$$f_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

You should be able to prove this with mathematical induction (see pp. 42–47 and Problem 2.3.32), but the more interesting question is, where did this formula come from? Think about this and other things that come up when you study the Fibonacci sequence.

1.3.18 An "ell" is an L-shaped tile made from three 1×1 squares, as shown below.



For what positive integers a, b is it possible to completely tile an $a \times b$ rectangle using only ells? ("Tiling" means that we cover the rectangle exactly with ells, with no overlaps.) For example, it is clear that you can tile a 2×3 rectangle with ells, but (draw a picture) you cannot tile a 3×3 with ells. After you understand rectangles, generalize in two directions: tiling ells in more elaborate shapes, tiling shapes with things other than ells.

1.3.19 Imagine a long $1 \times L$ rectangle, where L is an integer. Clearly, one can pack this rectangle with L circles of diameter 1, and no more. (By "pack" we mean that touching is OK, but overlapping is not.) On the other hand, it is not immediately obvious that $2L$ circles is the maximum number possible for packing a $2 \times L$ rectangle. Investigate this, and generalize to $m \times L$ rectangles.

1.4 How to Read This Book

This book is not meant to be read from start to finish, but rather to be perused in a "non-linear" way. The book is designed to help you study two subjects: problem solving methodology *and* specific